

Radiative damping: a case study

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Abstract. We are interested in the motion of a classical charge coupled to the Maxwell self-field and subject to a uniform external magnetic field, B_0 . This is a physically relevant, but difficult dynamical problem, to which contributions range over more than one hundred years. Specifically, we will study the Sommerfeld-Page approximation which assumes an extended charge distribution at small velocities. The memory equation is then linear and many details become available. We discuss how the friction equation arises in the limit of “small” B_0 and contrast this result with the standard Taylor expansion resulting in a second order equation for the velocity of the charge.

1 Introduction

Radiative damping must be taken into account in the design of synchrotron radiation sources [1]. Amongst other tasks, ultra strong lasers are expected to provide a quantitative test of radiative damping forces in the ultra relativistic regime [2]. For stellar jets and other highly accelerated beams of charged particles radiative friction plays a dominant role [1]. The ground work on radiative friction was accomplished already Abraham [3] and Lorentz [4], who undertook a vast effort to derive effective equations of motion for the charged particle starting from a model of an extended charged body coupled to the Maxwell field. Later on fully relativistic models followed. I refer to recent studies [5, 6] and to the monographs [7, 8, 9], in which prior work is put in perspective. To be clear, we will consider classical charges coupled to the classical electromagnetic field. When the interest is in the radiative decay of, say, an excited hydrogen atom, then a quantum version of the theory is required, which is beyond the scope of this note.

Granted a few exceptions, in all derivations of the effective equations of motion for the charged particle one uses Taylor expansion in a small parameter, which leads to a differential equation of second order in the velocity. In such a procedure it is never detailed how to match the true initial data with the initial conditions for the effective dynamics. In addition the second order differential equation generates spurious solutions which are a mere artifact of the approximation method. The case study presented here is so simple that both defects can be easily exhibited and remedied. While we discuss the extension to more physical models, we believe that a satisfactory derivation of the equations of radiative friction are still to be accomplished.

For the sake of discussion I prefer to fix a very concrete physical set-up, namely the motion of a charged particle subject to a uniform external magnetic field and coupled to its self-generated Maxwell field. The coupling to the self-field makes our dynamical problem difficult. To keep the presentation as simple as possible and to be able to provide a complete argument, we will study an oversimplified model known as Sommerfeld-Page equation. In the Sommerfeld-Page equation the charge distribution of the particle is taken to be uniform over a sphere of radius R and normalized to e . The charge distribution is supposed to be rigid and independent of velocity in the given inertial frame. Also small velocities are assumed. At the initial time $t = 0$ one has to specify position and velocity of the center of the charge distribution and the self-field. One natural choice for the self-field is the Coulomb field generated by the charge distribution at rest. An even more physical choice for the self-field would be the field as if the particle has been traveling with constant velocity v_0 for all $t \leq 0$. There are other choices and the initial Maxwell self-field is to some extent arbitrary. For sure, finite energy and some smoothness is required. On the other hand, only as one example, an incident plane wave would have to be taken into account explicitly in the Sommerfeld-Page equation.

With these assumptions the equations for the motion of the particle read

$$m_0 \frac{d}{dt} \mathbf{v}(t) = \frac{eB_0}{c} \mathbf{v}(t)^\perp + \frac{e^2}{12\pi c R^2} (\mathbf{v}(t - 2c^{-1}R) - \mathbf{v}(t)), \quad (1.1)$$

$$m_0 \frac{d}{dt} v_3 = \frac{e^2}{12\pi c R^2} (v_3(t - 2c^{-1}R) - v_3(t)). \quad (1.2)$$

Here m_0 is the mechanical (bare) mass of the particle, e the charge, and c the velocity of light. The magnetic field points along the 3-direction with magnitude B_0 . The velocity is decomposed in the component \mathbf{v} orthogonal and the component v_3 parallel to the magnetic field. If $\mathbf{v} = (v_1, v_2)$, then $\mathbf{v}^\perp = (-v_2, v_1)$. In the literature (1.1), (1.2) is referred to as Sommerfeld-Page equation [10, 11], see also Appendix A for more details on the approximations involved. We will discuss only (1.1), since (1.2) is automatically included as the special case $B_0 = 0$ in (1.1).

To streamline the notation, we set

$$\omega = \frac{eB_0}{m_0 c}, \quad \tau = \frac{2R}{c}, \quad \alpha = \frac{e^2}{3\pi m_0 c^3 \tau^2}. \quad (1.3)$$

Thus ω measures the strength of the external magnetic field and α the coupling strength to the self-field. (1.1) becomes then

$$\frac{d}{dt} \mathbf{v}(t) = \omega \mathbf{v}(t)^\perp + \alpha (\mathbf{v}(t - \tau) - \mathbf{v}(t)). \quad (1.4)$$

The natural dimensionless parameters will be $\omega\tau$ and $\alpha\tau$. (1.4) is a delay equation because the current change in velocity depends on the past history. Also the notion differential delay equation is very common. In statistical mechanics one uses the term memory equation, indicating again the dependence on the past. Eq. (1.4) has to be supplemented with the initial condition

$$\mathbf{v}(t) = \mathbf{u}(t), \quad 0 \leq t \leq \tau, \quad (1.5)$$

with some specified function $\mathbf{u}(t)$ encoding the interaction with the self-field up to time τ , an information which usually is not available in explicit form. In fact we will see that the precise details of $\mathbf{u}(t)$, except for the average and the end point $\mathbf{u}(\tau)$, play no role.

To obtain from (1.1) an effective differential equation seems to be an easy task. Since R is small, one Taylor expands in R up to second order and arrives at

$$\left(m_0 + \frac{e^2}{6\pi c^2 R}\right) \frac{d}{dt} \mathbf{w}(t) = \frac{eB_0}{c} \mathbf{w}(t)^\perp + \frac{e^2}{6\pi c^3} \frac{d^2}{dt^2} \mathbf{w}(t), \quad (1.6)$$

where we use $\mathbf{w}(t)$ to distinguish from the true solution $\mathbf{v}(t)$. Formally one expects that

$$\mathbf{v}(t) - \mathbf{w}(t) = \mathcal{O}(R). \quad (1.7)$$

The prefactor of $\dot{\mathbf{w}}(t)$ is regarded as effective mass,

$$m_{\text{eff}} = m_0 + \frac{e^3}{3\pi c^3 \tau} = (1 + \alpha\tau)m_0. \quad (1.8)$$

If one really wants $R \rightarrow 0$, following Dirac [12], one takes the double limit $R \rightarrow 0$, $m_0 \rightarrow -\infty$ with $m_{\text{eff}} > 0$ fixed. Then it is claimed that

$$\lim_{R \rightarrow 0} \mathbf{v}(t) = \mathbf{w}(t). \quad (1.9)$$

To fix a unique solution to (1.6) one has to prescribe $\mathbf{w}(0)$ and $\dot{\mathbf{w}}(0)$. On the other hand for (1.4) we need to specify a whole initial function $\mathbf{u}(t)$, $0 \leq t \leq \tau$. A complete derivation of (1.6), or (1.9), thus must give a prescription how to link $\mathbf{u}(t)$ to $\mathbf{w}(0)$ and $\dot{\mathbf{w}}(0)$. This part is missing and the Taylor expansion does not tell us.

In fact, the situation is even more complicated. The phase space for (1.6) is $\mathbb{R}^2 \times \mathbb{R}^2$. Embedded is the two-dimensional plane \mathcal{C}_ω with $0 \in \mathcal{C}_\omega$. If $(\mathbf{w}(0), \dot{\mathbf{w}}(0)) \in \mathcal{C}_\omega$, then the motion is stable and spirals towards 0. On the other hand for initial data off \mathcal{C}_ω , the motion diverges exponentially fast to infinity. Since such run away solutions have never been observed, I regard them as unphysical. But then one has to explain why for a broad class of initial $\mathbf{u}(t)$, $0 \leq t \leq \tau$, one always has initial data for (1.9) which lie in \mathcal{C}_ω . Again the Taylor expansion does not tell us.

For the Sommerfeld-Page equation a straightforward alternative to the Taylor expansion (1.6) is available. Eq. (1.4) is the standard and most elementary example of a linear differential-difference equation which is exhaustively covered in the literature [13, 14]. I will use here the conventional method for construction of solutions. Given the linear structure, it is rather surprising that so far the Sommerfeld-Page equation has been analyzed only rudimentarily [15, 16, 17, 18, 19, 20].

Our task is to study the solution to (1.4), (1.5). Roughly one expects that the synchrotron motion is damped and the particle will come to rest in the long time limit. In principle one could explore properties of (1.4) for all ω and α . But it is of advantage to keep the physical parameter range in mind. The classical electron radius is defined by the condition that the mechanical rest energy equals the electrostatic energy of the charge distribution. This yields $R \simeq 10^{-13}$ cm. Then $\alpha\tau = 10^{-5}$ and for a magnetic field of 10^6 Tesla one arrives at $\omega\tau = 10^{-2}$. The strongest laboratory fields are of order 10 Tesla. Thus it will be safe to eventually consider only the regime $0 < \alpha\tau \ll 1$, $|\omega\tau| \ll 1$.

Before entering into a more detailed analysis, let me summarize already now the main findings.

(1) *Zero external magnetic field, $B_0 = 0$.*

(i) If $m_{\text{eff}}/m_0 > 0$, then the motion is stable. On the other hand if $m_{\text{eff}}/m_0 < 0$, generically the solution increases exponentially. Expressed in dependence on m_0 , if $m_0 > 0$, then $m_{\text{eff}} > 0$, and the motion is stable. If $-e^2/3\pi c^3 \tau^2 < m_0 < 0$, then the motion is unstable but still $m_{\text{eff}} > 0$. If $m_0 < -e^2/3\pi c^3 \tau^2$, then $m_{\text{eff}} < 0$ and the motion is again stable.

(ii) If $m_{\text{eff}}/m_0 > 0$, then $\mathbf{v}(t) \rightarrow \bar{\mathbf{v}}$ exponentially fast as $t \rightarrow \infty$, where

$$\bar{\mathbf{v}} = (1 + \alpha\tau)^{-1}(\mathbf{u}(\tau) + \alpha \int_0^\tau ds \mathbf{u}(s)), \quad (1.10)$$

to say, there is a decay constant $\gamma_{\alpha\tau} < 1$ such that

$$|\mathbf{v}(n\tau) - \bar{\mathbf{v}}| \leq c_0(\gamma_{\alpha\tau})^n \quad (1.11)$$

with integer $n > 0$ and some constant c_0 . At $\alpha\tau = 0.278$, the decay constant $\gamma_{\alpha\tau} \approx 0.07$ and decreases rapidly as $\alpha\tau$ decreases to 0. At $\alpha\tau = -1$, the decay constant $\gamma_{\alpha\tau} \approx 0.25$.

(2) *Non-zero external magnetic field*, $B_0 \neq 0$, $|\omega\tau| \ll 1$, $m_0 > 0$.

(iii) If $m_0 > 0$, $|\omega\tau| \ll 1$, and $\omega\tau \neq 0$, then $\mathbf{v}(t) \rightarrow 0$ exponentially fast as $t \rightarrow \infty$ with an upper bound

$$|\mathbf{v}(n\tau)| \leq c_0 \left(1 - \frac{1}{2} \frac{\alpha\tau}{1 + \alpha\tau} \left(\frac{\omega\tau}{1 + \alpha\tau}\right)^2\right)^n. \quad (1.12)$$

(iv) For $m_0 > 0$ and at fixed R , if $|\omega\tau| \ll 1$, the solution to (1.4) is well approximated by

$$(1 + \alpha\tau) \frac{d}{dt} \mathbf{w}(t) = \omega \mathbf{w}(t)^\perp - \frac{1}{2} \omega^2 \tau \alpha\tau (1 + \alpha\tau)^{-2} \mathbf{w}(t). \quad (1.13)$$

More precisely, one can choose an initial condition $\mathbf{w}(0)$, such that $\mathbf{w}(0) = \bar{\mathbf{v}} + \mathcal{O}(\omega\tau)$ and

$$|\mathbf{v}(t) - \mathbf{w}(t)| \leq \mathcal{O}((\omega\tau)^2) \quad (1.14)$$

for all $t/\tau \geq |\log(\omega\tau)^2 / \log \gamma_{\alpha\tau}|$. Note that there is a slip in the initial velocity to $\bar{\mathbf{v}} + \mathcal{O}(\omega\tau)$. In the physical regime, instead of the more complicated memory equation (1.1), one can work with the friction differential equation (1.13), which in units of (1.1) reads

$$m_{\text{eff}} \frac{d}{dt} \mathbf{v}(t) = \frac{eB_0}{c} \mathbf{v}(t)^\perp - \frac{e^2}{6\pi c^3} \left(\frac{eB_0}{cm_{\text{eff}}}\right)^2 \mathbf{v}(t). \quad (1.15)$$

A detailed discussion and derivation of these results is presented in Sections 2 and 3, where we also explain the point charge limit $R \rightarrow 0$. In our context Landau and Lifshitz [21] provide a simple recipe how to proceed from (1.6) to arrive directly at (1.15). This point will be discussed in more detail in Section 4. Based on the novel insights from the Sommerfeld-Page equation we return to the full model, a charge coupled to its Maxwell self-field, and summarize of what has been achieved and what still has to be done, in our opinion.

2 Stability of the memory equation

For the study of the Sommerfeld-Page equation it will be convenient to use complex notation. Then $v(t) = v_1(t) + iv_2(t) \in \mathbb{C}$ and $u(t) = u_1(t) + iu_2(t) \in \mathbb{C}$. With this notation (1.4) reads

$$\frac{d}{dt}v(t) = -i\omega v(t) + \alpha(v(t - \tau) - v(t)) \quad (2.1)$$

and, according to (1.5),

$$v(t) = u(t) \quad \text{for} \quad 0 \leq t \leq \tau. \quad (2.2)$$

To find out $v(\tau + t)$, we integrate (2.1) on both sides. Then

$$v(\tau + t) = e^{-(\alpha + i\omega)t}u(\tau) + \alpha \int_0^t ds e^{-(\alpha + i\omega)(t-s)}u(s) = (K_\omega u)(t), \quad (2.3)$$

$0 \leq t \leq \tau$. We regard (2.3) as a linear map which out of v in the interval $[0, \tau]$ makes v in the interval $[\tau, 2\tau]$. We will call K_ω the Sommerfeld-Page operator. Note that K_ω is a non-symmetric operator. More details are explained in Appendix B. To obtain the solution $v(t)$ for all times one has to simply iterate the map (2.3) with the result

$$v(n\tau + t) = ((K_\omega)^n u)(t), \quad (2.4)$$

n integer and $0 \leq t \leq \tau$. Clearly, $(K_\omega u)(0) = u(\tau)$. If u is arbitrary, then $v(t + \tau)$, $0 \leq t \leq \tau$, is continuous and so is $v(t)$ for all $\tau \leq t < \infty$ by iteration.

The stability of solutions is determined by the eigenvalue problem for K_ω ,

$$K_\omega u = \lambda u. \quad (2.5)$$

Stable dynamics is equivalent to $|\lambda| \leq 1$ for all eigenvalues. Differentiating (2.5) with respect to t yields

$$\lambda \frac{d}{dt}u(t) = \alpha u(t) - \lambda(\alpha + i\omega)u(t), \quad (2.6)$$

which implies that the eigenvector u is an exponential function. Setting

$$u(t) = e^{zt/\tau}, \quad (2.7)$$

one concludes

$$\lambda = e^z \quad (2.8)$$

and $|\lambda| \leq 1$ precisely if $\Re z \leq 0$. Inserting in (2.6) one obtains

$$z = -i\omega\tau + \alpha\tau(e^{-z} - 1). \quad (2.9)$$

We will study case by case and, to simplify notation, switch to dimensionless variables. Since we want to allow $m_0 < 0$, it is convenient to set $\tau = 1$ for a while.

(i) *The case $\omega = 0$.* The eigenvalues of K_0 are determined by

$$z = \alpha(e^{-z} - 1). \quad (2.10)$$

We set $z = x + iy$, $x, y \in \mathbb{R}$, and first study solutions for which $\Im z = 0$. They are determined by

$$x + \alpha = \alpha e^{-x}. \quad (2.11)$$

$x = 0 = x_0(\alpha)$ is always a solution. Let us introduce the two critical points, α_+ , α_- . $\alpha_- = -1$ and α_+ is determined by

$$\alpha_+ e^{\alpha_+} = e^{-1}, \quad \alpha_+ \cong 0.278. \quad (2.12)$$

(2.11) has one further solution, $x_1(\alpha)$, provided $\alpha < 0$, where $x_1(\alpha) < 0$ for $\alpha_- < \alpha < 0$ and $x_1(\alpha) > 0$ for $\alpha < \alpha_-$. Thus the motion is unstable for $m_{\text{eff}}/m_0 < 0$.

Equating absolute value and phase of (2.10), one arrives at

$$(A) \quad (x + \alpha)^2 + y^2 = \alpha^2 e^{-2x}, \quad (2.13)$$

$$(P) \quad x + \alpha = -y \cot y. \quad (2.14)$$

The intersections of the curves (A) and (P) define the additional eigenvalues. (P) has branches labeled by $m \in \mathbb{Z}$ corresponding to $(m-1)\pi < y < m\pi$, $m \leq -1$, $-\pi < y < \pi$, $m\pi < y < (m+1)\pi$, $m > 0$. The 0-th branch has its apex at $x = -1 - \alpha$, $y = 0$ and converges to $\pm\pi$ as $x \rightarrow \infty$. The other branches are strictly increasing for $m > 0$, strictly decreasing for $m < 0$, and cover the full real line, see Fig.1.

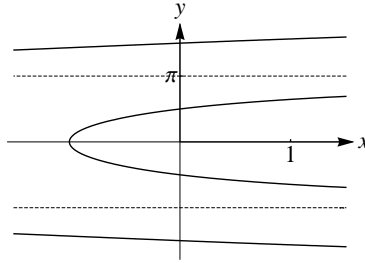


Figure 1: Curve (P) for $\alpha = 0$. Eigenvalues result by intersection with one of the curves from Fig. 2, resp. Fig. 3.

Next we discuss (A), which has the following solutions at $y = 0$, see Fig. 2 and 3: $x_0(\alpha) = 0$ is a solution for all α . $x_1(\alpha)$ and $x_2(\alpha)$ exists only for $\alpha < \alpha_+$. One has $x_1(\alpha) < x_2(\alpha) < 0$ for $0 < \alpha < \alpha_+$, $x_1(\alpha) < 0 < x_2(\alpha)$ for $\alpha_- < \alpha < 0$, and $0 < x_1(\alpha) < x_2(\alpha)$ for $\alpha < \alpha_-$. (A) has always a branch which diverges to $\pm\infty$ as $x \rightarrow -\infty$. Its apex is located at $(x_1(\alpha), 0)$ for $\alpha_- < \alpha < \alpha_+$ and at $(0, 0)$ for

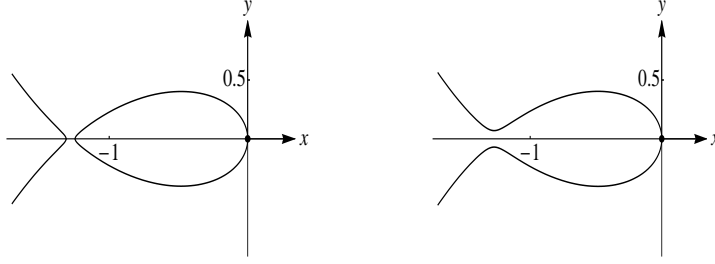


Figure 2: (a) the curve (A) for $\alpha = 0.27835 < \alpha_c^+$ and (b) for $\alpha = 0.279 > \alpha_c^+$. \bullet is an eigenvalue.

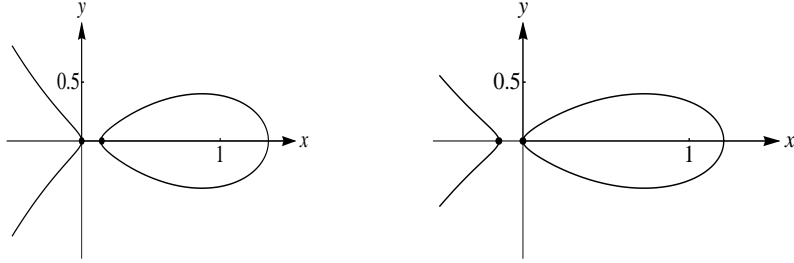


Figure 3: (a) the curve (A) for $\alpha = -1.07 < \alpha_c^-$ and (b) for $\alpha = -0.93 > \alpha_c^-$. \bullet is an eigenvalue.

$\alpha < \alpha_-$, $\alpha_+ < \alpha$. In addition (A) has a bubble with apices at $(x_2(\alpha), 0)$ and $(0, 0)$ for $\alpha_- < \alpha < \alpha_+$ and at $(x_1(\alpha), 0)$ and $(x_2(\alpha), 0)$ for $\alpha < \alpha_-$.

The 0-th branch of (P) never intersects (A). The branches $m \neq 0$ of (P) intersect the diverging branch of (A), which thus yields only stable eigenvalues. For $0 < \alpha < \alpha_+$ they satisfy $\Re z < -1 - \alpha_+$. The spectrum of eigenvalues of K_ω at $\omega = 0$ is plotted schematically in Fig. 4. The decay constant $\gamma_{\alpha\tau}$ is the radius of the smallest disk which contains all eigenvalues except for the eigenvalue 1. For $\alpha < \alpha_-$ the intersection points lie to the left but very close to the imaginary axis. Hence, the stability of motion is determined by the solutions to (2.11).

(ii) *The case $\omega \neq 0$, $|\omega| \ll 1$.* The eigenvalues are determined by

$$z = -i\omega + \alpha(e^{-z} - 1), \quad (2.15)$$

which induces a small shift. Of interest is the shift of the eigenvalue $\lambda = 1$, i.e. $z = 0$. Expanding at $z = 0$, one arrives at

$$z = -i\omega + \alpha(-z + \frac{1}{2}z^2), \quad (2.16)$$

which for small ω has the solution

$$z = -\frac{\alpha}{2(1+\alpha)^3}\omega^2 - i\frac{1}{(1+\alpha)}\omega. \quad (2.17)$$

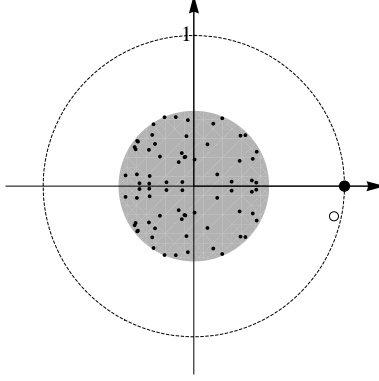


Figure 4: Schematic spectrum of K_ω . \bullet are eigenvalues for $\omega = 0$. The radius of the shaded disk is the decay constant. \circ indicates the shift of the eigenvalues by taking $\omega \neq 0$.

For $\alpha > 0$ the motion is stable, while for $-1 < \alpha < 0$ it is unstable. At $\alpha = -1$ there is a double degenerate eigenvalue 0, which splits into the solutions of $z^2 = -2i\omega$. Hence the unstable motion persists for $\alpha \leq -1$. For $|\omega| \ll 1$ and $\alpha > 0$, there is a single maximal eigenvalue, λ_{\max} , with the property

$$|\lambda_{\max}| = 1 - \frac{1}{2} \frac{\alpha}{1 + \alpha} \left(\frac{\omega}{1 + \alpha} \right)^2 + \mathcal{O}(\omega^3), \quad (2.18)$$

which implies a slow inward motion.

3 Long time behavior and comparison dynamics

These two topics are not specific for a charge coupled to the Maxwell field. The long time behavior is a first central task in the analysis of a dynamical system. Comparison dynamics is the issue of approximating the full dynamics through simplified, more tractable equations and typically requires to have a small parameter at one's disposal. The generic form of such a comparison is to first match the initial data then to provide error bars which specify how well and over what time scale a solution of the true dynamics is shadowed by the comparison dynamics. In the following we denote by $v(t)$ the true dynamics, as before, and the approximation by $w(t)$, both taking values in \mathbb{C} .

3.1 Long time asymptotics

To determine the long time behavior we need the maximal eigenvector and the spectral gap. As explained in more detail in the appendix, the natural space for the Sommerfeld-Page operator K_ω is the Hilbert space $\mathcal{H} = \mathbb{C} \oplus L^2([0, \tau], dt)$. A vector $(u_0, u) \in \mathcal{H}$ consists of the constant $u_0 = u(\tau)$ and the function $u(t)$, $0 \leq t \leq \tau$. The spectrum of K_ω acting on \mathcal{H} is denoted by $\sigma(K_\omega)$. As shown in Appendix B,

K_ω has the family of right eigenvectors $|g_\lambda\rangle$ and of left eigenvectors $\langle f_\lambda|$, $\lambda \in \sigma(K_\omega)$, such that

$$(K_\omega)^n = \sum_{\lambda \in \sigma(K_\omega)} \lambda^n (1 + \lambda^{-1} \alpha \tau)^{-1} |g_\lambda\rangle \langle f_\lambda|. \quad (3.1)$$

Explicit formulae for $|g_\lambda\rangle$, $|f_\lambda\rangle$ are given in (B.20) together with (B.15), (B.16) at $q = \lambda$.

We assume now $m_0 > 0$ and $\omega = 0$. Then 1 is the unique maximal eigenvalue with left eigenvector $|g_1\rangle = (1, 1)$ and right eigenvector $|f_1\rangle = (1, \alpha)$. Hence, with $P_1 = |1 + \alpha\rangle^{-1} |g_1\rangle \langle f_1|$,

$$(P_1 u)(t) = (1 + \alpha \tau)^{-1} \left(u(\tau) + \alpha \int_0^\tau ds u(s) \right) = \bar{v} \quad (3.2)$$

for all t , $0 \leq t \leq \tau$, which equals Eq. (1.10) in complex rather than 2-vector notation. The estimates for the decay constant are derived from the graphical representation of Section 2.

Turning to the case $\omega \neq 0$, the maximal eigenvalue λ_{\max} has the property that $|\lambda_{\max}| < 1$. Therefore $v(t)$ decays to 0 as $t \rightarrow \infty$ with decay constant $|\lambda_{\max}|$. This observation yields (1.12).

3.2 Approximation of small B_0 -field

We consider the case $|\omega \tau| \ll 1$, which physically corresponds to a “weak” magnetic field and assume $m_0 > 0$, i.e. $\alpha > 0$. Denoting the integer part by $[\cdot]$, it holds

$$v(t) = \left((K_\omega)^{[t/\tau]} u \right) (t - [t/\tau] \tau). \quad (3.3)$$

The maximal eigenvalue of K_ω is given by the solution to (2.15) close to $z = 0$. The decay constant is $\gamma_{\alpha\tau} \leq 0.07$. Therefore, with $\lambda = \lambda_{\max}$,

$$v(t) = \lambda^{[t/\tau]} (1 + \lambda^{-1} \alpha \tau)^{-1} |g_\lambda\rangle \langle f_\lambda| u (t - [t/\tau] \tau) + \mathcal{O}((\gamma_{\alpha\tau})^{[t/\tau]}). \quad (3.4)$$

For $|\omega \tau| \ll 1$, one obtains

$$\lambda^{[t/\tau]} \rightarrow \exp \left[\left(-i(1 + \alpha \tau)^{-1} \omega - \alpha \tau (2(1 + \alpha \tau)^3)^{-1} \omega^2 \tau \right) t \right] + \mathcal{O}((\omega \tau)^2). \quad (3.5)$$

The initial value is $\bar{v} + \mathcal{O}(\omega \tau)$. Finally, if $t/\tau \geq |\log(\omega \tau)^2 / \log \gamma_{\alpha\tau}|$, then $(\gamma_{\alpha\tau})^{[t/\tau]} \leq (\omega \tau)^2$. Thereby we have established (1.13) and (1.14).

3.3 The point charge limit

Following Dirac [12] we consider the limit

$$\tau \rightarrow 0 \quad \text{for fixed} \quad m_{\text{eff}} = m_0 + \frac{e^2}{3\pi c^3 \tau} > 0. \quad (3.6)$$

Then, for small τ ,

$$\alpha\tau = -1 - a\tau, \quad a = \frac{3\pi c^3}{e^2} m_{\text{eff}}, \quad (3.7)$$

and

$$\omega\tau = -b\tau^2, \quad b = \frac{3\pi c^3}{e^2} \frac{eB_0}{c}. \quad (3.8)$$

Since $\alpha\tau < -1$, but close to -1 , we now have to take into account the two maximal eigenvalues, λ_+ , λ_- , close to 1. The eigenvalue equation reads

$$z + i\omega\tau = \alpha\tau(e^{-z} - 1). \quad (3.9)$$

Setting $z = \kappa\tau$ one finds

$$\lambda_{\pm} = 1 + \tau\kappa_{\pm}, \quad \kappa_{\pm} = -a(1 \pm \sqrt{1 + i2b/a}). \quad (3.10)$$

In the limit $\tau \rightarrow 0$ it holds

$$(1 + \tau\kappa_{\pm})^{[t/\tau]} \rightarrow e^{\kappa_{\pm}t}. \quad (3.11)$$

In differential form, (3.11) agrees with Lorentz-Dirac equation (1.15).

Somewhat more tricky is the computation of the initial conditions. In the sum (3.1) only the eigenvalues λ_{\pm} contribute. All other eigenvalues satisfy $|\lambda| \leq 0.25$, independent of τ . We set $\lambda = 1 + \tau\kappa$ and omit the \pm index. For given $u(t)$, $0 \leq t \leq \tau$, one has to study

$$|g_{\lambda}\rangle \langle f_{\lambda}|u\rangle (1 + \alpha\tau\lambda^{-1})^{-1} \quad (3.12)$$

in the limit $\tau \rightarrow 0$.

Since, according to (B.16), (B.20),

$$|g_{\lambda}\rangle = (1, z^{-1} \exp[\alpha t(z^{-1} - 1) - i\omega t]), \quad (3.13)$$

with $\lambda = e^z$, one finds

$$\lim_{\tau \rightarrow 0} |g_{\lambda}\rangle = (1, 1). \quad (3.14)$$

The left eigenvector $|f_{\lambda}\rangle$ is given by

$$|f_{\lambda}\rangle = (1, f_{\lambda}(t)) \quad (3.15)$$

with $\lambda = e^z$ and

$$\begin{aligned} f_{\lambda}(t) = & -(1 - 2i\omega(z/\alpha))^{-1} 2i\omega e^{-(\alpha - i\omega)(\tau - t)} \\ & + (1 - 2i\omega(z/\alpha))^{-1} (\alpha/z) \exp[\alpha t(z^{-1} - 1) - i\omega(\tau - t)]. \end{aligned} \quad (3.16)$$

Hence the scalar product becomes

$$\langle f_{\lambda}|u\rangle = u(\tau) + \int_0^{\tau} ds u(s) f_{\lambda}(s)^*. \quad (3.17)$$

The contribution from the first summand of (3.16) is $\mathcal{O}(\tau^3)$. For the second term we expand as $u(t) = u(0) + u'(0)t$ and obtain the scalar product of (3.17) as $\frac{1}{2}\tau(u'(0) - \kappa u(0))$. Hence

$$\lim_{\tau \rightarrow 0} |g_\lambda\rangle \langle f_\lambda|u\rangle (1 + \alpha\tau\lambda^{-1})^{-1} = (2(\kappa - a))^{-1} (u'(0) - \kappa u(0)). \quad (3.18)$$

We conclude that in the point charge limit $v(t) \rightarrow w(t)$ as $\tau \rightarrow 0$ constrained by (3.6) and the limit motion is given by

$$w(t) = \sum_{\sigma=\pm} (2(\kappa_\sigma - a))^{-1} (u'(0) - \kappa_\sigma u(0)) e^{\kappa_\sigma t}. \quad (3.19)$$

κ_+ is the stable and κ_- the unstable mode. Unless there is a specific relation between $u(0)$ and $u'(0)$ the motion (3.19) is unstable. Dirac postulated his asymptotic condition, which in our context would single out the stable mode. From the point of view of the memory equation, as soon as $m_0 < 0$ with $m_{\text{eff}} > 0$, the motion is unstable. Thus it is not surprising that also the $\tau \rightarrow 0$ limiting motion is unstable. Of course, one could adjust the initial conditions to be orthogonal to $|f_{\lambda_-}\rangle$, thus maintaining a stable solution throughout the limiting procedure. Clearly, such a constraint would have to be justified by some additional reasoning beyond the Sommerfeld-Page equation.

4 The Landau-Lifshitz approximation

In the Landau-Lifshitz approximation the external \mathbf{B}_0 -field is considered to be small. To introduce the small parameter explicitly we substitute in (1.4) ω by $\varepsilon\omega$ with $0 < \varepsilon \ll 1$ a dimensionless parameter. To make up for the small field one has to consider times of order ε^{-1} . The rescaled time, again denoted by t , is in units of single revolutions. With both changes (1.4) becomes

$$\varepsilon \frac{d}{dt} \mathbf{v}(t) = \varepsilon \omega \mathbf{v}(t)^\perp + \alpha (\mathbf{v}(t - \varepsilon\tau) - \mathbf{v}(t)). \quad (4.1)$$

Taylor expanding in ε yields

$$(1 + \alpha\tau) \frac{d}{dt} \mathbf{v}(t) = \omega \mathbf{v}(t)^\perp + \frac{1}{2} \alpha \tau^2 \varepsilon \frac{d^2}{dt^2} \mathbf{v}(t). \quad (4.2)$$

The small parameter ε appears now in front of the highest derivative, which is the hallmark of singular perturbation theory [22]. The center manifold to order ε is singled out by re-substituting the leading order in (4.2) with the result

$$(1 + \alpha\tau) \frac{d}{dt} \mathbf{v}(t) = \omega \mathbf{v}(t)^\perp - \frac{1}{2} \varepsilon \alpha \left(\frac{\omega\tau}{1 + \alpha\tau} \right)^2 \mathbf{v}(t). \quad (4.3)$$

It seems that by a clever short-cut we have directly arrived at (1.13). If one compares with (1.6), then (4.3) corresponds to the weak B_0 -field approximation of the Lorentz-Dirac equation restricted to the stable plane \mathcal{C}_ω of physical solutions.

If one tries to control the error in the step from (4.1) to (4.2), one runs into the same difficulty as with the Lorentz-Dirac equation. Unstable solutions increase as $e^{t/\varepsilon}$ and it remains mysterious by which fine tuning of the initial conditions such runaways are avoided. But since we know from Section 3 that (4.3) is a controlled approximation to the true solution, we conclude that Taylor expanding (4.1) generates spurious solutions. They are an artifact of the approximation scheme used and not an intrinsic property of the Sommerfeld-Page equation.

5 Discussions and conclusions

Physically one would expect that the results from the case study are of a much greater generality. Of course the goal would be to treat a fully relativistic matter field coupled to the Maxwell field. Not so many rigorous results are available then and we propose to first have a look at the semi-relativistic Abraham model.

The dynamical variables are position \mathbf{q} , the velocity \mathbf{v} , and the self-fields $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$. In addition there is a uniform external magnetic field \mathbf{B}_0 . Then the coupled equations of motion read

$$c^{-1} \frac{\partial}{\partial t} \mathbf{B} = -\nabla \times \mathbf{E}, \quad c^{-1} \frac{\partial}{\partial t} \mathbf{E} = \nabla \times \mathbf{B} - \frac{e}{c} \varphi(\cdot - \mathbf{q}) \mathbf{v}, \quad (5.1)$$

$$\nabla \cdot \mathbf{E} = e\varphi(\cdot - \mathbf{q}), \quad \nabla \cdot \mathbf{B} = 0, \quad (5.2)$$

$$\frac{d}{dt} m_b \gamma \mathbf{v}(t) = \frac{e}{c} \mathbf{v}(t) \times \mathbf{B}_0 + \frac{e}{c} \int d\mathbf{x} \varphi(\mathbf{x} - \mathbf{q}(t)) (\mathbf{E}(\mathbf{x}, t) + \mathbf{v}(t) \times \mathbf{B}(\mathbf{x}, t)). \quad (5.3)$$

Here φ is the rigid charge distribution, $\varphi \geq 0$, φ radial and smooth, $\varphi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R$, $\int d\mathbf{x} \varphi(\mathbf{x}) = 1$. m_b is the mechanical bare mass and $\gamma(\mathbf{v}) = (1 - (\mathbf{v}/c)^2)^{-1/2}$. The initial position and velocity for the particle is $\mathbf{q}_0, \mathbf{v}_0, |\mathbf{v}_0/c| < 1$. For the self-field there is some freedom, as discussed already. To be specific we set

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_{\mathbf{v}_0}(\mathbf{x} - \mathbf{q}_0), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_{\mathbf{v}_0}(\mathbf{x} - \mathbf{q}_0), \quad (5.4)$$

where $\mathbf{E}_{\mathbf{v}_0}, \mathbf{B}_{\mathbf{v}_0}$ are the minimizers of the total, i.e. field plus particle, energy at given $\mathbf{q}_0, \mathbf{v}_0$. For an explicit formula we refer to [9], Eqs. (4.5) to (4.7).

As for the Sommerfeld-Page equation, the motion in the plane orthogonal to \mathbf{B}_0 can be decoupled. For notational simplicity, let us set $\mathbf{B}_0 = (0, 0, B_0)$. We assume

$$q_3(0) = 0, \quad v_3(0) = 0 \quad (5.5)$$

and require for the initial fields that

$$E_3(x_1, x_2, x_3) = -E_3(x_1, x_2, -x_3), \quad B_j(x_1, x_2, x_3) = -B_j(x_1, x_2, -x_3), \quad j = 1, 2. \quad (5.6)$$

Then from (5.1) – (5.3) it follows that these properties remain valid for all t . Thus for initial fields with the particular symmetry (5.6) the motion of the charge stays in the 1-2 plane.

We now follow the scheme of Section 4. In the fully relativistic setting this yields the effective equations of motion

$$m_0 \gamma \frac{d}{dt} \mathbf{v} = \frac{e}{c} (\mathbf{v} \times \mathbf{B}_0) + \frac{e^2}{6\pi c^3 m_0} \left(\frac{e}{m_0 c} \right)^2 [(\mathbf{v} \cdot \mathbf{B}_0) \mathbf{B}_0 - \mathbf{B}_0^2 \mathbf{v}] \quad (5.7)$$

valid for “small” \mathbf{B}_0 . Here m_0 is the rest mass of the particle. If one would aim at a precision of order $|\mathbf{B}_0|$, then the initial conditions for (5.7) are simply $\mathbf{q}_0, \mathbf{v}_0$.

The semi-relativistic character of Abraham model is merely reflected in a velocity dependent effective mass which differs from the one of a relativistic particle. The relevant computation can be found in [9], Chapter 4.1. With this difference one can still work out the re-substitution as indicated in Section 4. The expressions become unwieldy and there is no particular reason to display them here. The structure of the effective equations of motion is similar to (5.7).

For our particular initial condition, if $\mathbf{B}_0 = 0$, then the particle would simply continue to travel with velocity \mathbf{v}_0 . Thus one might guess that the effective equations of motion (5.7) should be solved with initial conditions $\mathbf{v}(0) = \mathbf{v}_0$. As discussed at length in our case study, to have an approximation which includes friction and thus allows for an error $\mathcal{O}(\mathbf{B}_0^2)$ only, one expects that the initial conditions must be corrected slightly by a term of order $|\mathbf{B}_0|$.

The details of such a program are carried out in [23]. As emphasized already, upon Taylor expanding one arrives at a comparison dynamics which has runaway solutions making difficult the precise control of errors. The best we could accomplish is an estimate of the difference in energies of the true and comparison dynamics. A pointwise comparison of the trajectories is missing.

More tractable is the issue of the long time asymptotics. Under certain conditions it is proved that, in case $\mathbf{B}_0 = 0$, $\lim_{t \rightarrow \infty} \mathbf{v}(t) = \bar{\mathbf{v}}$ [24, 25, 26]. Also, if the charge is subject to an external electrostatic potential, then in for long times the particle comes to rest at a critical point of the potential [27]. On the other side, for a uniform external magnetic field $\mathbf{B}_0 \neq 0$ the energy balance used in [27] is not strong enough to control the long time behavior and it remains as an open problem to establish that the charge comes to rest in the limit $t \rightarrow \infty$.

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A Appendix: Derivation of the Sommerfeld-Page equation

In standard derivations of the Sommerfeld-Page equation, one expands the solution of the full Abraham model in R as a small parameter retaining only the terms linear in \mathbf{v} . In [9], Chapter 7, we provide a somewhat different perspective. No originality is claimed. Starting from the full Abraham model one can write down an exact memory equation for the motion of the charge. There are two terms: one term results from the initial conditions of the Maxwell field and a memory term, which contains only the self-generated field. The charge distribution is assumed to be radially symmetric and supported in a ball of radius R , but otherwise arbitrary, and the Maxwell field is assumed to be co-moving. Then memory refers to times up to $2R/c$ into the past and after a time of order $2R/c$ the initial Maxwell field no longer influences the motion of the charge. No expansion is employed. If one now specializes to a charge distribution uniformly over a sphere of radius R and retains in the memory term only contributions linear in \mathbf{v} , then one arrives at the Sommerfeld-Page equation (1.1).

B Appendix: The Sommerfeld-Page operator

We provide a more detailed analysis of the Sommerfeld-Page operator K_ω which reads

$$(K_\omega u)(t) = e^{-(\alpha+i\omega)t} u(\tau) + \alpha \int_0^t ds e^{-(\alpha+i\omega)(t-s)} u(s), \quad 0 \leq t \leq \tau. \quad (\text{B.1})$$

The eigensolutions to (B.1),

$$K_\omega u_\lambda = \lambda u_\lambda, \quad (\text{B.2})$$

are of the form

$$u_\lambda(t) = e^{zt/\tau}, \quad \lambda = e^z. \quad (\text{B.3})$$

Here $\lambda \in \sigma(K)$, the spectrum of K , and λ is implicitly determined through the solutions of

$$z + i\omega\tau = \alpha\tau(e^{-z} - 1), \quad (\text{B.4})$$

which coincides with (2.9). Assuming the initial condition u as the linear superposition

$$u(t) = \sum_{\lambda \in \sigma(K_\omega)} c_\lambda u_\lambda(t), \quad (\text{B.5})$$

one finds

$$(K_\omega)^n u(t) = \sum_{\lambda \in \sigma(K_\omega)} c_\lambda \lambda^n u_\lambda(t). \quad (\text{B.6})$$

For the particular case $\omega = 0$ and $1 + \alpha\tau > 0$, $\lambda = 1$ is the unique maximal eigenvalue. We then split as

$$(K_0)^n u(t) = c_1 + \sum_{\lambda \in \tilde{\sigma}(K_\omega)} c_\lambda \lambda^n u_\lambda(t), \quad (\text{B.7})$$

$\tilde{\sigma}(K) = \sigma(K) \setminus \{1\}$. Therefore

$$|(K_0)^n u(t) - c_1| \leq \left(\sum_{\lambda \in \tilde{\sigma}(K)} |c_\lambda| \right) \gamma^n \quad (\text{B.8})$$

with the decay constant $\gamma = \max\{|\lambda|, \lambda \in \tilde{\sigma}(K)\}$. While this provides the desired exponential decay, the constant in front of γ^n might be large. In particular, one would like to have a bound which is expressed directly in terms of the initial u and not in the expansion coefficients c_λ .

If in (B.1) u is assumed to be continuous, then the dual space is the space of finite complex measures over $[0, \tau]$. After one application of the dual operator K_ω^* the measure has a continuous part $u(t)dt$, $0 \leq t \leq \tau$, and a δ -function at τ . Thus the natural space for K_ω is $\mathbb{C} \oplus L^2([0, \tau], dt) = \mathcal{H}$. For $(c, u) \in \mathcal{H}$, c corresponds to the complex weight of the δ -function at τ and u to $u(t)$, $0 \leq t \leq \tau$. With this representation the operator can be written as

$$K_\omega = \begin{pmatrix} \beta & \langle f| \\ |g\rangle & A \end{pmatrix}, \quad (\text{B.9})$$

where $\beta = e^{-(\alpha+i\omega)\tau}$, $g(t) = e^{-(\alpha+i\omega)t}$, $f(t) = e^{-(\alpha-i\omega)(\tau-t)}$, and the kernel of A , $A(t, s) = \chi(s \leq t) \alpha e^{-(\alpha+i\omega)(t-s)}$.

To estimate $(K_\omega)^n$ for large n we use the resolvent of K_ω . As verified directly by integration, one finds

$$(q - A)^{-1}(t, s) = q^{-1} \delta(t - s) + q^{-2} \chi(s \leq t) \alpha e^{-(\alpha+i\omega)(t-s)} e^{q^{-1} \alpha(t-s)} \quad (\text{B.10})$$

for all $q \in \mathbb{C} \setminus \{0\}$. K_ω is a one-dimensional perturbation of A . Hence

$$(q - K_\omega)^{-1} = \begin{pmatrix} (q - \beta)^{-1} & 0 \\ 0 & (q - A)^{-1} \end{pmatrix} + D(q)^{-1} \begin{pmatrix} (q - \beta)^{-1} \langle f|(q - A)^{-1} g \rangle & \langle f|(q - A)^{-1} \\ (q - A)^{-1} |g\rangle & (q - A)^{-1} |g\rangle \langle f|(q - A)^{-1} \end{pmatrix}, \quad (\text{B.11})$$

where

$$D(q) = q - \beta - \langle f|(q - A)^{-1} g \rangle. \quad (\text{B.12})$$

The scalar product is obtained to

$$\langle f|(q - A)^{-1} g \rangle = e^{-(\alpha+i\omega)\tau} e^{q^{-1} \alpha \tau} - e^{-(\alpha+i\omega)\tau} \quad (\text{B.13})$$

and hence

$$D(q) = q - e^{-(\alpha+i\omega)\tau} e^{q^{-1}\alpha\tau}. \quad (\text{B.14})$$

Setting $q = e^z$ one observes that $D(q) = 0$ is equivalent to (B.4) and (B.11) holds whenever $q \in \mathbb{C} \setminus \sigma(K_\omega)$.

From (B.10), using $g(t)$, $f(t)$ as below (B.9),

$$(q - A)^{-1} f(t) = -2i\omega(1 - 2i\omega q \alpha^{-1})^{-1} e^{-(\alpha-i\omega)(\tau-t)} + \alpha q^{-1}(1 - 2i\omega q \alpha^{-1})^{-1} e^{-(\alpha-i\omega)(\tau-t)} e^{q^{-1}\alpha(\tau-t)} = f_q(t), \quad (\text{B.15})$$

$$(q - A)^{-1} g(t) = q^{-1} e^{-(\alpha+i\omega)t} e^{q^{-1}\alpha t} = g_q(t). \quad (\text{B.16})$$

The poles of the resolvent define the spectrum $\sigma(K_\omega)$. There is a pole at $q = 0$, which does not contribute to the spectral representation of K_ω . There is an apparent pole at $q = \beta$, which can be lifted since the numerator vanishes at $q = \beta$. Finally there are the poles resulting from the zeros of D ,

$$\sigma(K_\omega) = \{\lambda | D(\lambda) = 0\}, \quad (\text{B.17})$$

which has been shown already to be in agreement with (B.4). The residue is computed from

$$D'(\lambda) = 1 + \tau \lambda^{-1}. \quad (\text{B.18})$$

Altogether, with \mathcal{C}_λ a small contour around λ , one obtains

$$P_\lambda = \oint_{\mathcal{C}_\lambda} dq (q - K_\omega)^{-1} = (1 + \alpha \tau \lambda^{-1})^{-1} |g_\lambda\rangle \langle f_\lambda| \quad (\text{B.19})$$

for $\lambda \in \sigma(K_\omega)$ with

$$|g_\lambda\rangle = (1, g_\lambda(t)), \quad |f_\lambda\rangle = (1, f_\lambda(t)). \quad (\text{B.20})$$

With these preparations we obtain

$$K_\omega = \sum_{\lambda \in \sigma(K_\omega)} \lambda (1 + \alpha \tau \lambda^{-1})^{-1} |g_\lambda\rangle \langle f_\lambda|. \quad (\text{B.21})$$

There are no Jordan block's since the projectors in (B.21) are one-dimensional. As a consequence

$$(K_\omega)^n = \sum_{\lambda \in \sigma(K_\omega)} \lambda^n (1 + \alpha \tau \lambda^{-1})^{-1} |g_\lambda\rangle \langle f_\lambda|. \quad (\text{B.22})$$

For the particular case $\omega = 0$, $1 + \alpha \tau > 0$, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} (K_0)^n u(t) &= g_1(t) \langle f_1 | u \rangle \\ &= (1 + \alpha \tau)^{-1} \left(u(\tau) + \alpha \int_0^\tau ds u(s) \right), \end{aligned} \quad (\text{B.23})$$

as claimed in (1.10).

To find the rate of convergence, under the same conditions, we subtract P_1 and use

$$\|K_0^n - P_1\| \leq \sum_{\lambda \in \tilde{\sigma}(K)} |\lambda|^n (1 + \alpha\tau\lambda^{-1}) \|g_\lambda\| \|f_\lambda\|. \quad (\text{B.24})$$

Computing the norms in (B.24) one uses

$$|\lambda + \alpha\tau|^{-1} |\lambda| \|g_\lambda\| \|f_\lambda\| = |\lambda + \alpha\tau|^{-1} |\lambda| (1 + |\lambda + \lambda^* - 2|\lambda|^2|^{-1} (1 - |\lambda|^2)) \leq (\alpha\tau)^{-1} \quad (\text{B.25})$$

for all $\lambda \in \tilde{\sigma}(K)$. The last inequality can be checked directly using that $\lambda = e^z$ satisfies (2.15). Hence

$$\|K_0^n - P_1\| \leq \sum_{\lambda \in \tilde{\sigma}(K)} |\lambda|^n (\alpha\tau)^{-1} \cong (\lambda_{\max})^n (\alpha\tau)^{-1}. \quad (\text{B.26})$$

In particular,

$$\int_0^\tau ds (K_0^n u(s) - \bar{u})^2 \leq (\lambda_{\max})^{2n} (\alpha\tau)^{-2} \int_0^\tau ds u(s)^2, \quad (\text{B.27})$$

which is a more explicit bound than (B.8).

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